MMAT5010 Home Test 2 **30 Apr 2021 at 2:00pm**

Important Notice:

♣ The answer paper **Must be submitted before 01 May 2021 at 2:00pm.**
♠ The answer paper MUST BE sent to the CU Blackboard.
✠ The answer paper **Must include your name and student ID.**

Answer ALL Questions

1. **(15 points)**
Let X be a normed space and let x_0 be a non-zero vector in X.

   (a) Show that there is an element $f \in X^*$ such that $f(x_0) = 1$.

   (b) Show that there is a non-identity bounded linear operator T from X to itself such that $Tx_0 = x_0$ and $\|T\| = 1$.

2. **(15 points)**
Let $1 \leq p < \infty$. Let $X := \{x \in \ell_p : \sum_{n=1}^{\infty} |nx(n)|^p < \infty\}$. Define a linear operator $T : X \to \ell_p$ by

$$Tx(n) := nx(n) \quad \text{for } x \in X \text{ and } n = 1, 2, ...$$

   (a) Is T a bounded operator? (Explain !)

   (b) Is T an isomorphism from X onto ℓ_p?

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3. **(15 points)**

Let H be a complex Hilbert space. Let (x_n) be a sequence in H. We say that (x_n) weakly converges to an element x in H if $\lim_n f(x_n) = f(x)$ for all $f \in H^*$. In this case, we call x a weak limit of the sequence (x_n).

(a) Show that if a sequence (x_n) is weakly convergent, then its weak limit is unique.

(b) Show that a sequence (x_n) converges weakly to an element $x \in H$ and $\lim \|x_n\| = \|x\|$ if and only if the sequence (x_n) converges to x in norm.

4. **(15 points)**

Let H be a Hilbert space. Let P and Q be the orthogonal projections on the closed subspaces M and N of H respectively.

(a) Show that $P - Q$ is an orthogonal projection if and only if $N \subseteq M$.

(b) Show that if $P - Q$ is an orthogonal projection, then $(P - Q)(H) = M \cap N^{\perp}$.

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